Computer Analysis of Gradually Tapered Waveguide with Arbitrary Cross Sections

S. S. SAAD, J. B. DAVIES, MEMBER, IEEE, AND O. J. DAVIES, MEMBER, IEEE

Abstract—**This short paper presents a general computer analysis of gradually tapered waveguide with arbitrarily shaped cross sections. The technique combines coupled-mode theory with numerical methods for solving the uniform waveguide problem. The coupling coefficients are computed by using eigenvalues and eigenfunctions obtained numerically. The mode amplitudes are obtained either by numerical solution of a set of differential equations or from a closed-form solution.**

The applicability of this technique is illustrated by the analysis and measurement of two transitions. It is shown that theoretical prediction of coupled-mode amplitudes is reliable for gradual tapers where the flare angle is small. For large flare angles, more rigorous coupled-mode theory has to be employed.

I. INTRODUCTION

The gradually tapered waveguide with arbitrarily shaped cross sections is of importance because of its use in various waveguide systems. A transducer converting the rectangular TE_{10} waveguide mode to the circular TE_{01} mode has, for instance, immediate use in the trunk waveguide systems which are being developed in many countries at present [1]. The general problem of gradually tapered guided structures also occurs at the lower frequencies (for example, tapered microstrip or slot line) and higher frequencies (for certain optical coupling between fibres and thin films).

A general analysis of such tapers, based on coupled-mode theory and using field and circuit concepts, has been given by Solymar [2]. The three-dimensional problem is replaced by a two-dimensional one of uniform waveguide leading to a set of $2N$ simultaneous ordinary linear differential equations in $2N$ unknowns, N being the number of modes propagating along the taper. The coefficients in these equations are associated with the eigenvalues and eigenfunctions of the uniform waveguide with cross section equal to that of the taper at any particular point.

The use of this technique has generally been restricted to problems where analytical eigenfunctions are available for the different cross sections along the taper. This short paper describes a technique that breaks the aforementioned restriction and so solves the problem of the general taper with arbitrary cross sections. This is made possible by taking advantage of numerical methods for solving the problem of the arbitrary uniform waveguide [3]. By substituting the computed eigenvalues and eigenfunctions into Solymar's equations, the problem of the arbitrary taper is then solved numerically. This short paper gives details of work that was briefly reported by the authors [4], [5]. Some work with similar objectives, but different techniques and applications, has been reported by Schindler [6].

II. THEORETICAL ANALYSIS AND NUMERICAL TECHNIQUE

A. The Coupled-Mode Equations

The generalized telegraphist's equations were transformed by Solymar [2] into a set of $2N$ differential equations. The amplitudes of the forward and backward traveling waves, A_i^+ and A_i^- in the lossless waveguide taper of Fig. 1 are

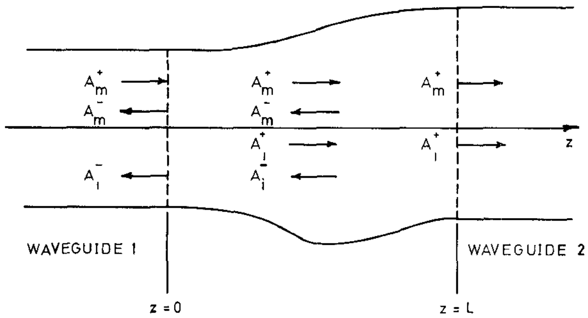

Fig. 1. Waveguide transition.

$$\frac{dA_i^+}{dz} + j\beta_i A_i^+ = -\frac{1}{2}\frac{d(\ln K_i)}{dz}A_i^- + \sum_{p=1}^{N}(S_{ip}^+A_p^+ + S_{ip}^-A_p^-),$$

$$\frac{dA_i^-}{dz} - j\beta_i A_i^- = -\frac{1}{2}\frac{d(\ln K_i)}{dz}A_i^+ + \sum_{p=1}^{N}(S_{ip}^-A_p^+ + S_{ip}^+A_p^-),$$

$$i = 1, 2, \cdots N \quad (1)$$

where $\beta_i(z)$ and $K_i(z)$ are the propagation coefficient and the wave impedance of the ith mode. They are related as follows:

$$K_{[i]} = \frac{\omega\mu}{\beta_{[i]}} \quad K_{(i)} = \frac{\beta_{(i)}}{\omega\varepsilon} \quad (2)$$

where the brackets denote a TE mode and the parentheses a TM mode. The taper itself should be smooth, that is, if the equation of the taper surface is $f(x,y,z) = 0$, then both $f(x,y,z)$ and its z derivative should be continuous along z.

The coupling terms on the right-hand side of (1) can be identified as due respectively to change with z of wave impedance of individual modes, and to the direct geometric effect (the taper coupling of different modes via the boundary conditions). Explicit forms have been obtained by Solymar [2] for the forward and backward coupling coefficients S_{ip}^+ and S_{ip}^- between the ith and pth modes and will not be repeated here. The coefficients $S_{ip}^\pm$ depend on the fields and cutoff frequencies of the ith and pth modes, and so their computation depends on the numerical solution of the transverse Helmholtz equation. The most promising numerical methods are reviewed and discussed by Davies [3] and Ng [7]. In the course of this work, three numerical methods were found useful, namely, the finite difference [8], the finite element [9], and the polynomial approximation method [10]. All three methods are available as program packages, and solve the Helmholtz equation by approximating the scalar field ψ by some function over the waveguide cross section. Different techniques are used to set up and to solve the standard form of the eigenvalue matrix equation.

B. Solution for the Waveguide Transition.

Having computed $S_{ip}^\pm$ and β_i for relevant modes along the waveguide, it is possible to calculate the mode amplitudes (and therefore the S matrix of the transducer) by small coupling theory or by numerical methods. By small coupling theory, Solymar [2] obtains

$$A_m^+(z) = A_0 \exp\left(-j\int_0^z \beta_m\, dz\right) \quad (3)$$

$$A_m^-(0) = -A_0\int_0^L \left\{S_{mm}^- - \frac{1}{2}\frac{d(\ln K_m)}{dz}\right\} \exp\left(-j2\int_0^z \beta_m\, dz\right) dz \quad (4)$$

Manuscript received November 4, 1974; revised November 1, 1976. This work was supported under Contract by the British Post Office [18].
S. S. Saad was with the University College, London, England. He is now with the Electrical Engineering Laboratory, National Research Centre, Dokki, Cairo, Egypt.
J. B. Davies and O. J. Davies are with the Department of Electronic and Electrical Engineering, University College, Torrington Place, London, England.

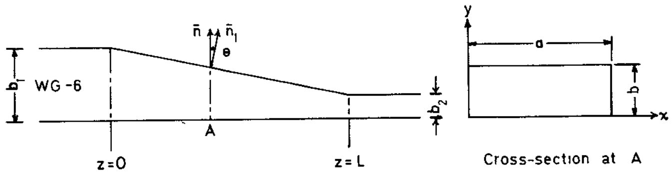

Fig. 2. Linear taper between two rectangular waveguides carrying the TE_{01} mode.

with boundary conditions

$$A_m^-(L) = A_m^+(0) = 0. \tag{5}$$

Note that an erroneous term in Solymar's expression has been removed to give (4), although the only effect is on the phase of $A_m^-(0)$.

For the numerical solution of (1), standard routines that solve simultaneous differential equations include the Runge–Kutta method and a variety of methods of predictor–corrector type [11]. Since (1) described a two-point boundary value problem, it does not lend itself directly to any of these methods, and a few introductory steps must be taken before they can be applied. This disadvantage is absent in the new technique developed by Denman [12] who employed a recursive algorithm, along with the method of invariant imbedding.

It is essential to compute the mode eigenvalue and eigenfunctions at sufficiently close cross sections for the accurate solution of (1) or of (4). Because of the oscillatory term in the integrands of (4), it is essential that the spacing between successive cross sections be sufficiently small compared with the guide wavelength (see Section IV). However, since the taper is gradual, β_i and $S_{ip}^{\pm}$ were calculated at a reasonable number of cross sections and polynomial curve fitting [13], [14] was used to give them as continuous functions of z.

It is important to note that these solutions at the various cross sections are evaluated only once for any geometry. Analysis of the taper performance at different frequencies requires further calculation of just the closed form [(4)] or numerical solution of the differential equation system in z alone [(1)].

III. Numerical Applications and Experimental Verification

By utilizing the technique presented in Section II, the solution of an arbitrary taper is now possible. In order to check the technique, two practical tapers were examined.

A. The Linearly Tapered Rectangular Waveguide

This type of taper was chosen for two reasons. Firstly, the transverse eigenvalue problem has an analytical solution. Hence the coupled-mode theory can be dissociated from the problem of numerical solution of the transverse eigenfunctions. Secondly, experimental results obtained by Young [15] are available.

The linear taper considered by Young is shown in Fig. 2. It connects the standard rectangular waveguide WG-6, operating in L band (1.12–1.70 GHz), to a guide having the same width but with reduced height. The design parameters of the taper are $b_1 = 3.25$ in, $b_2 = 0.40$ in, $a = 6.50$ in, and $L = 19.392$ in.

Assuming a pure TE_{10} mode is incident at $z = 0$, and taking advantage of the symmetry, single-mode operation is obtained over the operating frequency band. Furthermore, the wave impedance is constant; thus the set of (1) reduces to

$$\frac{dA_m^{\pm}}{dz} \pm j\beta_m A_m^{\pm} = S_{mm}^-(z) A_m^{\pm} \tag{6}$$

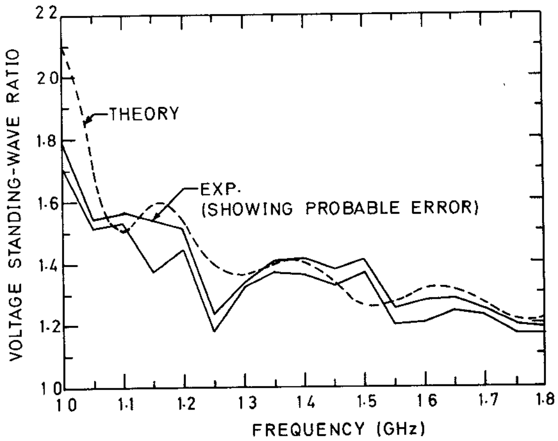

Fig. 3. The VSWR/frequency curves for the linear taper; comparison between theory and experiment.

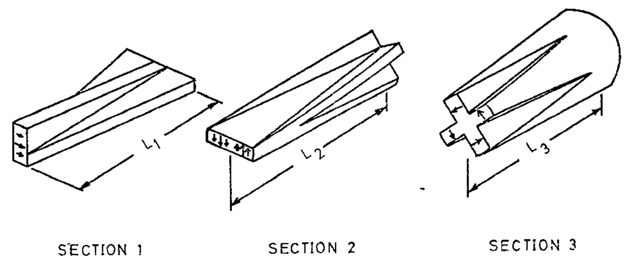

Fig. 4. Sections of the Marié transducer.

and the approximate solution (4) for the reflection of the TE_{10} mode becomes

$$A_m^-(0) = -A_0 \int_0^L S_{mm}^-(z) \exp(-j2\beta_m z)\, dz \tag{7}$$

resulting in the following expression for the reflected wave amplitude:

$$A_m^-(0) = -\tfrac{1}{2} A_0 e^{ju_1} \{\mathrm{Ci}(u_1) - \mathrm{Ci}(u_2) + j\mathrm{Si}(u_1) - j\mathrm{Si}(u_2)\} \tag{8}$$

where $\mathrm{Ci}(u)$ and $\mathrm{Si}(u)$ are the cosine and sine integrals, and

$$u_1 = 2\beta b_1/\tan\theta \quad u_2 = 2\beta b_2/\tan\theta. \tag{9}$$

The two simultaneous differential equations (6) were also solved numerically, using the standard subroutine DLBVP [16] which combines the method of adjoint equations with a predictor–corrector technique. The results agree with those of the approximate expression (8) within 5 percent for the voltage reflection coefficient. The numerical results are shown in Fig. 3 along with the experimental results of Young [15]. There is reasonable agreement between theory and experiment, remembering that the reflections caused by the abrupt discontinuity at the ends of the taper have not been accounted for in the theory.

B. The Marié Mode Transducer

This taper [17] converts the rectangular TE_{10} mode into the circular TE_{01} mode through three distinct sections which are illustrated in Fig. 4 together with the associated field patterns of the main mode.

A detailed theory of the transducer together with an optimized design are given in [18]. A pair of transducers was made and tested in band 33–50 GHz. The model matches the standard rectangular waveguide WG 23 (0.224×0.112 in) to the circular waveguide of 13-mm diameter. The transducer length was chosen

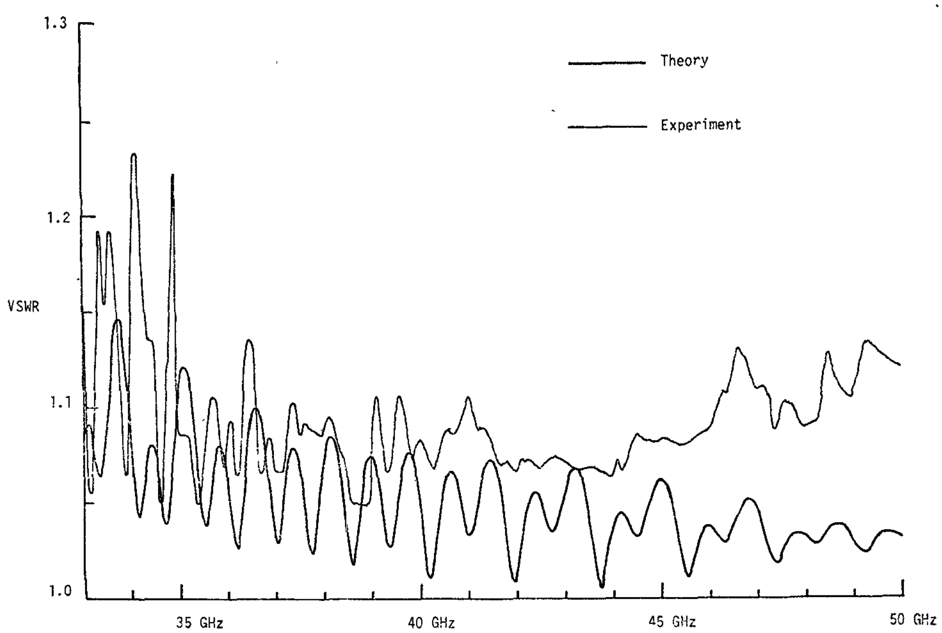

Fig. 5. VSWR/frequency curves of a Marié transducer.

as 7.75 in ($2\frac{1}{4}$ guide wavelength at midband) by an optimization scheme [18] (using the aforementioned analysis technique) to give minimum insertion loss.

Because of the many symmetries of the Marié transducer, the TE_{10} rectangular mode couples only to TE_{01}, TE_{41}, $TE_{02}\cdots$ modes of the circular guide [18]. Computations of the eigenvalues along the length of the transducer showed that single-mode operation is obtained over the frequency band 33–40 GHz. The return loss is then a convenient parameter to compare theory and experiment. Having computed the eigenvalues along the transducer, to give $\beta_m(z)$ and $K_m(z)$, the backward coupling coefficient of the main mode S_{mm}^- is calculated. Equation (4) then gives the complex reflection coefficient.

The measured VSWR is shown in Fig. 5 together with that calculated using (4). The agreement between theory and experiment can be considered satisfactory, bearing in mind the small VSWR and the fabrication difficulties; the transducer is 32 wavelengths long at 50 GHz and surface tolerances were ± 0.02 mm.

IV. Discussion and Conclusions

In the case of the Marié transducer, satisfactory agreement between theory and experiment was obtained. There is perhaps better agreement for the linear taper (Section III-A) which can be attributed to its higher mechanical accuracy. Compared with the direct numerical solution of the set of differential equations (1), the approximate expression (4) for A_m^- is clearly easier to calculate. For the examples treated in this short paper (Sections III-A and III-B), the solutions of (1) and (4) were in good agreement for a reflection coefficient less than 0.2.

The technique (of combining numerical methods with Solymar's theory) has been checked only with single-mode structures. However, it is believed to be a representative test since the small coupling to the reflected wave of the main mode is mathematically no different from the small coupling to forward or backward traveling waves in a multimode system [2].

It can be concluded that the new technique is a valid and effective one for the analysis and design of any gradual taper with arbitrarily shaped cross sections. An advantage of this technique is the insight it gives to transducer operation and hence leading to improved design (e.g., identification of modes, trapped modes, and local reflection coefficient). A worthwhile application was the design of the Marié transducer resulting in an insertion loss averaging only 0.2 dB over the 33–50-GHz band [5].

The present technique seemingly has no serious or fundamental limitations. The range of validity of the small coupling theory used in this work can be extended to cover steep tapers. For instance, Stevenson [19] has given exact coupled-mode equations for a general taper, and his approach could be used to derive exact equations in $A_i^{\pm}$ corresponding to (1).

It would be prohibitively inefficient to develop one computer program that can deal with all shapes of tapers. However, a general routine can be easily developed to calculate the S matrix of any taper if the transverse eigenfunction and eigenvalue at various z values are input data. These functions should be computed by separate routines chosen for the convenience of the cross-sectional shapes. Computer time is mainly spent in the analysis of the cross sections. From experience, it was found sufficient to analyze one cross section per guide wavelength for gradual tapers. But again, the computer time varies considerably according to cross-sectional shape, accuracy demanded, and numerical method used [3], [7].

It should also be noted that for a given transducer, the aforementioned computer analysis of the cross-sectional modes provides data used at all frequencies. Once this analysis has been performed, evaluation of the scattering matrix at various frequencies requires solution *only* of the z-dependent problem, viz. computation of the explicit equation (4) when the taper is gradual. For the Marié transducer of Section III-B, the computing time is about 1 s for each frequency on an IBM 360/65.

The technique presented in this short paper can, in principle, be applied to dielectric rod, optical, or acoustic waveguide tapers of arbitrary cross sections.

References

[1] "Trunk telecommunications by guided waves," *IEE Conference Publication*, no. 71, London, 1970.
[2] L. Solymar, "Spurious mode generation in nonuniform waveguide," *IRE Trans. Microwave Theory Tech.*, vol. MTT-7, pp. 379–383,

July 1959.
[3] J. B. Davies, "Review of methods for numerical solution of the hollow-waveguide problem," *Proc. IEE* (London), vol. 119, pp. 33–37, Jan. 1972.
[4] J. B. Davies, O. J. Davies, and S. S. Saad, "Computer analysis of gradually tapered waveguide of arbitrary cross-sections," *Electron. Lett.*, vol. 9, pp. 46–47, Feb. 1973.
[5] S. S. Saad, J. B. Davies, and O. J. Davies, "Computer analysis and design of gradually tapered waveguide with arbitrary cross sections," presented at the URSI Symposium on Electromagnetic Wave Theory, London, July 9–12, 1974.
[6] G. Schindler, "Waveguide transitions especially for excitation of the H_{11} mode in circular waveguide," (in German), *Arch. Elektron. Übertragung, 27*, pp. 348–354, July/Aug. 1973.
[7] F. L. Ng, "Tabulation of methods for the numerical solution of the hollow waveguide problem," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 322–329, Mar. 1974.
[8] J. B. Davies and C. A. Muilwyk, "Numerical solution of uniform hollow waveguides with boundaries of arbitrary shape," *Proc. IEE* (London), vol. 113, pp. 277–284, Feb. 1966.
[9] P. Silvester, "A general high-order finite-element waveguide analysis program," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 204–210, Apr. 1969.
[10] R. M. Bulley, "Analysis of the arbitrarily shaped waveguide by polynomial approximation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1022–1028, Dec. 1970.
[11] L. Fox, Ed., *Numerical Solution of Ordinary and Partial Differential Equations.* New York: Pergamon Press, 1962, part 1.
[12] E. D. Denman, *Coupled Modes in Plasmas, Elastic Media and Parametric Amplifiers.* New York: Elsevier, 1970, ch. 3 and 6.
[13] A. T. Berztiss, "Least-squares fitting of polynomials to irregularly spaced data," *SIAM Review*, vol. 6, pp. 203–227, July 1964.
[14] C. W. Clenshaw and J. G. Hayes, "Curve and surface fitting," *J. Inst. Math. Applics.*, vol. 1, pp. 164–183, June 1965.
[15] L. Young, "Practical design of a wide-band quarter-wave transformer in waveguide," *Microwave J.*, vol. 6, pp. 76–79, Oct. 1963.
[16] "Subroutines LBVP and DLBVP," in System/360 Scientific Subroutine Package, Version III, IBM Programmer's Manual No. 360A-CM-03X, IBM, 1968, pp. 350–360.
[17] G. R. P. Marié, British Patent no. 805 510, filed Oct. 22, 1956, published Dec. 10, 1958.
[18] S. S. Saad, J. B. Davies, and O. J. Davies, "The rectangular TE_{10} to circular TE_{01} mode transducer of the Marié type," Final Report, British Post Office Research Contract No. 523283, Nov. 1973.
[19] A. F. Stevenson, "General theory of electromagnetic horns," *J. Appl. Physics*, vol. 22, pp. 1447–1460, Dec. 1951.

Letters

Computation of the Hecken Impedance Function

J. H. CLOETE

The Dolph–Chebyshev impedance function derived by Klopfenstein [1] has discontinuities at the taper ends which introduce unwanted effects in certain applications. The Hecken impedance function [2] is not optimum in the Dolph–Chebyshev sense, but achieves matching without impedance steps. For any bandwidth and maximum magnitude of reflection coefficient in the passband, the Hecken taper is only slightly longer than the optimum taper [2]. Hecken's near-optimum taper is therefore an attractive alternative to the optimum taper when impedance discontinuities are undesirable.

The equation for the near-optimum impedance function contains a transcendental function $G(B,\xi)$ which is tabulated in Hecken's paper. The function is given by

$$G(B,\xi) = \frac{B}{\sinh B}\int_0^{\xi} I_0\{B\sqrt{1-\xi'^2}\}\, d\xi'$$

where $I_0(z)$ is the modified Bessel function of the first kind and zero order.

Instead of using the tables, $G(B,\xi)$ may be computed recursively as

$$G(B,\xi) = \frac{B}{\sinh B}\sum_{k=0}^{\infty} a_k b_k$$

where

$$a_0 = 1 \quad a_k = \frac{B^2}{4k^2}\, a_{k-1}$$

$$b_0 = \xi \quad b_k = \frac{\xi(1-\xi^2)^k + 2kb_{k-1}}{2k+1}.$$

The derivation is based on the method described by Grossberg [3] and is not given here.

Manuscript received November 12, 1976.
The author is with the Council for Scientific and Industrial Research, Pretoria, South Africa.

References

[1] R. W. Klopfenstein, "A transmission line taper of improved design," *Proc. IRE*, vol. 44, pp. 31–35, Jan. 1956.
[2] R. P. Hecken, "A near-optimum matching section without discontinuities," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 734–739, Nov. 1972.
[3] M. A. Grossberg, "Extremely rapid computation of the Klopfenstein impedance taper," *Proc. IEEE*, vol. 56, pp. 1629–1630, Sept. 1968.

Synthesis of Certain Transmission Lines Employed in Microwave Integrated Circuits

RAYMOND CRAMPAGNE AND GRATIA KHOO

With a quasi-TEM approximation, the characteristic parameters of numerous structures used as hyperfrequency microelectronics transmission lines can be calculated with the aid of conformal mapping. Simple theoretical formulas are rarely used since they bring into play the function $K(k)/K'(k)$ where $K(k)$ is the complete elliptic integral of the first type, $K'(k)$ its complementary function, and k its argument.

Some geometrical configurations which can be treated are shown in Fig. 1(a)–(c). This method is particularly interesting since expressions of k (argument of elliptic integral) as a function of geometric dimensions are often simple.

The infinite dielectric thickness hypothesis made in certain cases is, in general, justified by the spacing between conductors. Although this method is surprisingly simple accompanied by a large application domain, it has been put aside by many research workers. Instead, sophisticated numerical methods like those of finite differences and finite elements [1] have been preferred. These methods are applicable for the analysis of transmission lines but not for the synthesis. Moreover, they do not lead to

Manuscript received April 27, 1976; revised September 10, 1976.
The authors are with the Microwave Laboratory, National Polytechnic Institute of Toulouse, Toulouse, France.